On the Lagrangian in a Covariant Field Theory

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Abstract

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1. As has been noted (see 1), the serious difficulties in General Relativity (GR) require a reexamination of the foundations of classical field theory. Lagrange formalism, as generally used, is not rigorous enough. We lack even an invariant definition of the Lagrange field structure on the manifold. Rigorous necessary and sufficient conditions for the covariance of the Lagrangian field theory are not known. We claim to show this on the example of GR.

2. Let V be a Riemannian space with a group G of general covariant transformations and ϕ be a tensor field on V. Let us construct the Lagrangian action functional invariant under the group G:

$$S\left[\phi\right] = \int L\left(\phi, \partial_\mu \phi, \partial_\nu \partial_\mu \phi, ...\right) d\Omega \tag{1}$$

where L is a Lagrangian, $\partial_\mu \phi$ is a partial derivative of ϕ and $d\Omega$ is an invariant element of volume in V.

It is generally accepted that invariance of action S (accordingly, Lagrangian L) is the condition, necessary and sufficient, to secure the covariance of the whole field theory. However, this is not correct, as even the "canonical momentum" of the field

$$\pi^\mu = \frac{\partial L}{\partial\left(\partial_\mu \phi\right)} \tag{2}$$

turns out to be noncovariant (for simplicity's sake, only first partials are indicated).

This leads, in addition, to noncovariance of the canonical energy-momentum tensor

$$T_\mu^\nu = \pi^\nu \partial_\mu \phi - \delta_\mu^\nu L \tag{3}$$

where $\left(\delta_\mu^\nu\right)$ is the Kronecker Tensor. This situation occurs in GR, where the selection of the Hilbert Lagrangian in the form $L = R\left(g_{\alpha\beta}, \partial_\mu g_{\alpha\beta}, \partial_\nu \partial_\mu g_{\alpha\beta}\right)$,

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where R is a Riemannian curvature scalar and $g_{\alpha\beta}$ is a metric tensor of the Riemannian space V^4, results in noncovariance of the energy-momentum of the gravitational field. This is one of the aspects of the well known Energy Problem.

Thus, invariance of the action functional is a necessary but not sufficient condition to secure the covariance of the theory. This misunderstanding has come about due to the absence of a rigorous definition of an invariant Lagrange field structure on a manifold. In this note we shall try to provide this structure.

3. Let M be a differentiable manifold with affine connection ∇ and metric g. Let ϕ be a differentiable tensor (or spinor) field on M and $J^k(\phi)$ be the k-th order jet bundle over M. Let us construct a morphism $L : J^k(\phi) \to R$.

The triplet $\langle M, \phi, L\rangle$ will be called a Lagrange Field Structure on differentiable manifold M with Lagrangian L. The action functional is defined as usual

$$S[\phi] = \int L d\Omega \tag{4}$$

where $d\Omega$ is an invariant element of volume.

Let x be a chart with field of definition W, then

$$L = L\left(\phi, \nabla_{\mu_1}\phi, \nabla_{\mu_2}\nabla_{\mu_1}\phi, ..., \nabla_{\mu_k}...\nabla_{\mu_1}\phi\right) \tag{5}$$

where ∇_{μ_k} is the covariant derivative with respect to affine connection ∇.

Thus, the Lagrangian must be an invariant function of covariant arguments, i.e. it must depend on the fields and their covariant derivatives. This is obviously the condition, necessary and sufficient, to secure invariance of the Lagrange Field Structure.

Consequently, in GR we have to write $L = L\left(g_{\alpha\beta}, \nabla_\mu g_{\alpha\beta}, \nabla_\nu\nabla_\mu g_{\alpha\beta}\right)\left(\nabla_\mu g_{\alpha\beta} \neq 0!\right)$.

Affine connection ∇ is a gauge field with respect to the field ϕ.

Now, all the quantities appearing in the theory, including the canonical momentum

$$\pi^\mu = \frac{\partial L}{\partial\left(\nabla_\mu\phi\right)}$$

and, accordingly, the energy-momentum tensor

$$T^\nu_\mu = \pi^\nu\nabla_\mu\phi - \delta^\nu_\mu L$$

will be covariant.

4. Correct definition of the Lagrange Field Structure is particularly important for GR where, as it had been previously noted, an incorrect form of the Lagrangian leads to the Energy Problem and obviously to other difficulties. Hopefully the revision of GR in the sense suggested will alleviate the necessity of attempting to resolve the arising difficulties with alternative theories of gravitation.

References

[1]Poltorak, A. "Towards a Covariant Theory of Gravitation." GR9 Conference Abstracts, Jena, 2, 516, 1980. (Available on http://www.arXiv.org as gr-qc/0403050)